Patterns in Probability

A Random Walk

												
L6	L5	L4	L3	L2	L1	O	R1	R2	R3	R4	R5	R6

A man stands at O and tosses a coin. If the coin lands on Heads he moves one step to the right (in the R-Direction). If the coin lands on Tails, he moves one step to the left (in the L-Direction)

1. How many different positions could the man occupy after 1 toss of the coin? Name these positions and find the probability of him being in each of them.

The man can end up at L1 or at R1 and p(L1) = ½ p(R1) = ½

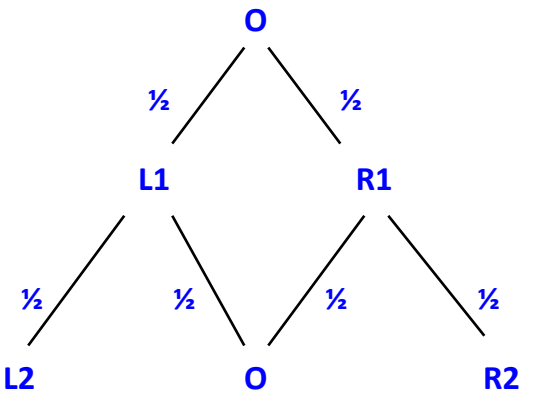
The man tosses the coin for a second time, with the same rule applying (Heads means he moves one step to the right, Tails means one step to the left).

2. How many different positions could the man occupy after 2 tosses of the coin? Name these positions and find the probability of him being in each of them.

The man can end up at L2 or at O or at R2. The probabilities are given by:

p(L2) = ½ x ½ = ¼ p(O) = (½ x ½) + (½ x ½) = ½ p(R2) = ½ x ½ = ¼

A tree diagram might be useful here:

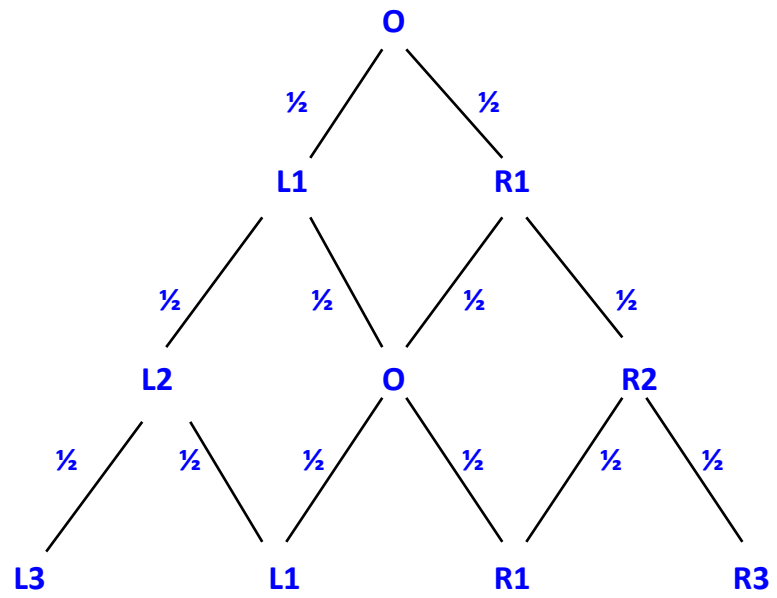


Final probability ¼ ¼ + ¼ ¼

The man continues with the coin-tossing and the moving.

3. How many different positions could the man occupy after 3 tosses of the coin? Name these positions and find the probability of him being in each of them (note: there may be more than one way in which the man can end up at a given position)

The man can end up at L3 or at L1 or at R1 or at R3. The probabilities are given via:



Probability	1/8	3/8	3/8	1/8

4. How many different positions could the man occupy after 4 or 5 tosses of the coin? Name these positions and find the probability of him being in each of them.

After 4 tosses, the man can end at L4, L2, O, R2 or R4

The probabilities are:
p(L4) = 1/16, p(L2) = 4/16, p(O) = 6/16, p(R2) = 4/16 and p(R4) = 1/16

After 5 tosses, the man can end at L5, L3, L1, R1, R3 or R5

The probabilities are:
p(L5) = 1/32, p(L3) = 5/32, p(L1) = 10/32, p(R1) = 10/32, p(R3) = 5/32 and p(R5) = 1/32

5. Summarise your results in an appropriate diagram or chart and comment on any patterns that are evident.

A Chart summarizing the probabilities:

	L5	L4	L3	L2	L1	O	R1	R2	R3	R4	R5
n=1					1/2		1/2				
n=2				1/4		2/4		1/4			
n=3			1/8		3/8		3/8		1/8		
n=4		1/16		4/16		6/16		4/16		1/16	
n=5	1/32		5/32		10/32		10/32		5/32		1/32

(another possible chart is an extension of the tree diagram above)

Patterns that are evident:

- **The denominator of the n^{th} row fractions is 2^n**
- **The numerator of any fraction is the sum of the two immediately above**
- **The fractions in any row sum to 1**
- **There is symmetry about the central vertical axis**
- **The numerator of the fraction is essentially the number of possible paths that take you to that ending position**

6. Based on your results, or otherwise, how many positions could he occupy after n tosses of the coin?

For n tosses, there appear to be n + 1 possible final positions

7. Prove or justify your answer to question 7

Imagine a step to the left is -1 and a step to the right is +1. After several left/right steps we hit a total which is (+1) times all the right steps plus (-1) times all the left steps

After n steps, the possible running totals are:

n, (n – 2), (n – 4), (n – 6), …… (-n+2), -n

There are n + 1 numbers in this list

8. What is the most likely finishing position after 4 tosses?

The most likely finishing position is at O
This is evident from the table above; it has the greatest single probability.

9. What is the most likely finishing position after 6 tosses?

The most likely finishing position is again at O

This is evident from the table below; it has the greatest single probability.

	L6	L5	L4	L3	L2	L1	O	R1	R2	R3	R4	R5	R6
n=1						1/2		1/2					
n=2					1/4		2/4		1/4				
n=3				1/8		3/8		3/8		1/8			
n=4			1/16		4/16		6/16		4/16		1/16		
n=5		1/32		5/32		10/32		10/32		5/32		1/32	
n=6	1/64		6/64		15/64		20/64		15/64		6/64		1/64

10. What is the most likely finishing position after an even number of tosses?

The most likely finishing position is at O

11. Prove/Justify your answer to Q10.

As has been remarked upon, the numerator of the fraction essentially tells us how many ways we can end up at that final destination.

Without any loss of generality, let's look at the case where n=6.

To end at R6, the path has to be R1R2R3R4R5R6 – this is the only path that takes you there

To end at R4, there are 6 possible paths:

R1 R2 R3 R4 R5 R4
R1 R2 R3 R4 R3 R4
R1 R2 R3 R2 R3 R4
R1 R2 R1 R2 R3 R4
R1 0 R1 R2 R3 R4
L1 0 R1 R2 R3 R4

Similarly, there are 15 possible ways of ending at R2 and 20 ways of ending at O

The most likely ending position is thus O.
And this situation will be repeated for all (even) values of n

We can continue the table above indefinitely, of course, and we have said that any fraction's numerator is the sum of the two numerators above. Given the pattern up to n=6 (as in the table), and given that the "new" end probabilities for the next values of n will have numerators of 1, the numerators on any row will increase from the end to the middle (then decrease from the middle to the end). Thus the biggest numerator (and therefore the biggest probability) will be in the central position (for O).